Severi varieties and branch curves
of abelian surfaces of type $(1,3)$

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Introduction

Let (A, L) be a principally polarized abelian surface of type (d_1, d_2). The linear system $|L|$ defines a map $\varphi_L : A \to \mathbb{P}^*_{d_1 d_2 - 1} = \mathbb{P}(H^0(L)^*)$. The geometry of this map is well understood except for $(d_1, d_2) = (1, 3)$ (see [CAV]). In this case φ_L is a $6 : 1$ covering of $\mathbb{P}^*_2$ with branch curve $B \subset \mathbb{P}^*_2$ of degree 18. The first problem is to study the curve B. The main result of this paper is that for a general abelian surface (A, L) of type $(1, 3)$ the curve B is irreducible, admits 72 cusps, 36 nodes or tacnodes, each tacnode counting as two nodes, 72 flexes and 36 bitangents. The main idea of the proof is to use the fact that for a general (A, L) of type $(1, 3)$ the discriminant curve $V \subset \mathbb{P}_2 = |L| = \mathbb{P}(H^0(L))$, coincides with the closure of the Severi variety $V_{L,1}$ of curves in $|L|$ admitting a node, and is dual to the curve B in the sense of projective geometry. We investigate V and B via degeneration to a special abelian surface.

In more detail the contents of the paper is as follows. In section 1 some well known facts about families of nodal curves on surfaces are recalled and proved for the case of an abelian surface. In section 2 we consider curves of genus 2 in a linear system $|L|$ as above. There are only finitely many such curves and we prove that they have only ordinary singularities. In section 3 we introduce and study the incidence curve. We explain how it is related to B and V and compute its numerical characters. Section 4 is devoted to a detailed analysis of a special case: we consider the abelian surface $A = E \times E$, where E is an elliptic curve, suitably polarized by a line bundle of type $(1, 3)$, and we describe completely the curves B and V. In the final section 5 we take up the general case: we degenerate a general (A, L) of type $(1, 3)$ to the surface considered in §4 and, using the analysis made in that special case, we prove our main result.

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§1. Generalities on Severi varieties

We recall a few known facts on families of nodal curves on an algebraic surface.

Let S be a nonsingular projective connected algebraic surface, $L \in \mathrm{Pic}(S)$ such that the complete linear system $|L|$ has a nonsingular and connected general member. The linear system $|L|$ is contained in a larger system of divisors, the *continuous system* defined by L, which will be denoted by $\{L\}$ in what follows. By definition $\{L\}$ is the connected component of Hilb^S containing $|L|$. If $X \subset S$ is a curve belonging to the system $\{L\}$ we will denote by $[X]$ the point of $\{L\}$ parametrizing X.

For each integer $\delta \geq 0$ denote by $V_{L,\delta} \subset |L|$ the functorially defined locally closed subscheme which parametrizes the family of all curves in $|L|$ having δ nodes and no other singularity (see [W]); they will be called δ-*nodal curves* . We likewise denote by $V_{\{L\},\delta} \subset \{L\}$ the analogous closed subscheme of $\{L\}$ parametrizing δ-nodal curves.

It is customary to call *regularity* the condition of being nonsingular of codimension δ for $V_{L,\delta}$ or $V_{\{L\},\delta}$.

Let $X \in V_{L,\delta}$ and let $N \subset S$ be the scheme of its nodes, which of course consists of δ distinct reduced points. We have an exact commutative diagram of sheaves on S:

$$
\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \mathcal{O}_S & = & \mathcal{O}_S & & \\
 & & \sigma_X \downarrow & & \downarrow \sigma_X & & \\
0 \to & \mathcal{I}_{N/S} \otimes L & \to & L & \to & L \otimes \mathcal{O}_N & \to 0 \\
 & & \downarrow & & \downarrow & & \|\wr \\
0 \to & \mathcal{I}_{N/X} \otimes N_X & \to & N_X & \to & T_X^1 & \to 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
\end{array}
$$

where $N_X = L \otimes \mathcal{O}_X$ is the normal sheaf of X in S, and T_X^1 is the first cotangent sheaf of

X (see [W]). Consider the following diagram obtained by taking cohomology of it:

$$
\begin{array}{ccccc}
0 \to & \dfrac{H^0(\mathcal{I}_{N/S}\otimes L)}{\langle\sigma_X\rangle} & \to & \dfrac{H^0(L)}{\langle\sigma_X\rangle} & \xrightarrow{\ h\ } & H^0(L\otimes\mathcal{O}_N) \\
& \cap & & \cap & & \| \\
0 \to & H^0(\mathcal{I}_{N/X}\otimes N_X) & \to & H^0(N_X) & \to & H^0(T^1_X) \\
& \downarrow & & \downarrow & & \\
& H^1(\mathcal{O}_S) & = & H^1(\mathcal{O}_S) & & \\
& \downarrow & & \downarrow & & \\
& H^1(\mathcal{I}_{N/S}\otimes L) & & 0 & & \\
& \downarrow & & & & \\
& H^1(\mathcal{I}_{N/X}\otimes N_X) & & & & \\
& \downarrow & & & & \\
& H^2(\mathcal{O}_S) & & & & \\
& \downarrow & & & & \\
& 0 & & & &
\end{array}
$$

where the zero at the left bottom is because $H^2(\mathcal{I}_{N/S}\otimes L)=0$. We have identifications:

$$
\frac{H^0(\mathcal{I}_{N/S}\otimes L)}{\langle\sigma_X\rangle}=\mathbf{T}_{[X]}(V_{L,\delta})
$$

$$
\frac{H^0(L)}{\langle\sigma_X\rangle}=\mathbf{T}_{[X]}(|L|)
$$

$$
H^0(\mathcal{I}_{N/X}\otimes N_X)=\mathbf{T}_{[X]}(V_{\{L\},\delta})
$$

and

$$
H^0(N_X)=\mathbf{T}_{[X]}(\{L\})
$$

where $\mathbf{T}_{[X]}(-)$ denotes the tangent space of $-$ at $[X]$. Therefore the above diagram is

read as:

$$
\begin{array}{ccccccc}
0 \to & \mathbf{T}_{[X]}(V_{L,\delta}) & \to & \mathbf{T}_{[X]}(|L|) & \xrightarrow{\;h\;} & H^0(L \otimes \mathcal{O}_N) \\
& \cap & & \cap & & \| \\
0 \to & \mathbf{T}_{[X]}(V_{\{L\},\delta}) & \to & \mathbf{T}_{[X]}(\{L\}) & \to & H^0(T_X^1) \\
& \downarrow & & \downarrow & & \\
& H^1(\mathcal{O}_S) & = & H^1(\mathcal{O}_S) & & \\
& \downarrow & & \downarrow & & \\
& H^1(\mathcal{I}_{N/S} \otimes L) & & 0 & & \\
& \downarrow & & & & \\
& H^1(\mathcal{I}_{N/X} \otimes N_X) & & & & \\
& \downarrow & & & & \\
& H^2(\mathcal{O}_S) & & & & \\
& \downarrow & & & & \\
& 0 & & & &
\end{array}
\tag{1}
$$

and all the maps between tangent spaces are the differentials of the corresponding inclusions. Since we have $\mathrm{codim}_{|L|}(V_{L,\delta}) \le \delta$, we see that

Proposition 1.1.

$$V_{L,\delta} \text{ is regular at } [X] \Leftrightarrow h \text{ is surjective}$$

$$\Leftrightarrow\; N \text{ imposes independent conditions to } |L| \Leftrightarrow H^1(\mathcal{I}_{N/S} \otimes L) = 0$$

Note that if S is a regular surface then $V_{L,\delta} = V_{\{L\},\delta}$ and the vertical inclusions are equalities.

Let A be an abelian surface and L an ample line bundle of type (d_1, d_2) on A. We call $d := d_1 d_2$ the *degree* of L. According to Riemann-Roch the linear system $|L|$ is of dimension $d - 1$, which we will assume to be ≥ 2. We always assume that $|L|$ has no fixed component. This means that the polarized abelian variety (A, L) is not of the form $(E_1 \times E_2, p_1^* L_1 \otimes p_2^* L_2)$ with elliptic curves E_1 and E_2 and line bundles L_1 of degree 1 on E_1 and L_2 of degree d on E_2. According to [CAV] Proposition 10.1.3 the general member of $|L|$ is a smooth irreducible curve on A of genus $d + 1$. We have

$$\dim(\{L\}) = \dim(|L|) + 2 = d + 1$$

because $\{L\}$ consists of a two-dimensional family of translates of $|L|$. More precisely, denoting by $\hat{A}$ the dual of A, we have a surjective morphism:

whose fibres are linear systems. For the same reason

$$\dim(V_{\{L\},\delta}) = \dim(V_{L,\delta}) + 2$$

The curve X has arithmetic genus

$$p_a(X) = \frac{1}{2}L^2 + 1 = d + 1$$

and $N_X = \omega_X$ is its dualizing sheaf. Therefore

$$h^0(N_X) = p_a(X) = d + 1 = dim(\{L\})$$

and it follows that $\{L\}$ is nonsingular of dimension $d + 1$ at X.

The differential of ψ at X is the coboundary map

$$H^0(N_X) \to H^1(\mathcal{O}_A)$$

which is surjective because $H^1(L) = 0$. It follows that ψ is smooth at $[X]$ with fibre $\psi^{-1}(\psi(X)) = |L|$.

In order to compute $\mathbf{T}_{[X]}(V_{\{L\},\delta})$ consider the normalization map $\nu : C \to X$. Then we have an isomorphism

$$H^0(X, \mathcal{I}_{N/X} \otimes N_X) = H^0(X, \mathcal{I}_{N/X} \otimes \omega_X) \cong H^0(C, \omega_C)$$

Then

$$\dim[\mathbf{T}_{[X]}(V_{\{L\},\delta})] = h^0(C, \omega_C) = g$$

where g is the geometric genus of C, which is given by the formula

$$g = p_a(X) - \delta + c - 1$$

where c is the number of connected components of C. Since

$$\dim[V_{\{L\},\delta}] \geq p_a(X) - \delta$$

we have:

Proposition 1.2. *Let (A, L) be as above. Then $V_{\{L\},\delta}$ is regular at $[X]$ if and only if X is irreducible.*

In particular we see that the conditions for regularity of $V_{L,\delta}$ (Prop. 1.1) and of $V_{\{L\},\delta}$ (Prop. 1.2) are different. Nevertheless we have the following:

Proposition 1.3 *In the same situation of Prop. 1.2, assume that X is reducible. Then $V_{L,\delta}$ is not regular at $[X]$.*

Proof. Diagram (1) shows that if $h^1(\mathcal{I}_{N/X} \otimes N_X) \geq 2$ then $h^1(\mathcal{I}_{N/S} \otimes L) \neq 0$ and $V_{L,\delta}$ is not regular by Prop. 1.1. Since from the above analysis it follows that $h^1(\mathcal{I}_{N/X} \otimes N_X)$ is equal to the number of irreducible components of X, the conclusion follows. $\qquad \square$

Remark. Note that Propositions 1.1 and 1.2 apply as well to the case when S is a K3-surface. In this case $V_{L,\delta} = V_{\{L\},\delta}$ and we obtain the following combination of the two propositions:

Proposition 1.4. *Let S be a K3-surface, L a globally generated line bundle on S such that $\dim(|L|) \geq 2$, and $X \in V_{L,\delta}$. Then $V_{L,\delta}$ is regular at $[X]$ if and only if X is irreducible.*

In the sequel we will be especially interested in $V_{L,1}$ and in its closure $\overline{V}_{L,1}$. From the assumptions made on L it follows that $|L|$ is base point free and therefore Proposition 1.1 implies that $V_{L,1}$ is regular (i.e. nonsingular of codimension 1) everywhere.

We will also consider the *discriminant locus* $V \subset |L|$ (see §3 for its scheme-theoretic definition) which parametrizes all singular curves of $|L|$. Of course we have $\overline{V}_{L,1} \subset V$ but the inclusion is proper in general (see §4).

§2 Singularities of curves of genus 2 on abelian surfaces.

Let (A, L) be an abelian surface of type (d_1, d_2). In this section we show that every curve of genus 2 in $|L|$ admits at most ordinary singularities. For this we need the following result on principally polarized abelian surfaces which is certainly well-known. Not knowing a reference we give a proof.

Proposition 2.1. *Let $(J, \mathcal{O}_J(C))$ denote a principally polarized abelian surface. If translates $t_x^* C$ and $t_y^* C$ of the curve C do not have a component in common, they intersect exactly in 2 distinct points.*

Proof. J is either a product of 2 elliptic curves $E_1 \times E_2$ and $C = E_1 \times \{0\} + \{0\} \times E_2$ or the Jacobian of a smooth curve C of genus 2. The assertion being obvious in the product case, we may assume that C is smooth of genus 2 containing $0 \in J$. Translating and passing to an algebraically equivalent line bundle we see that it suffices to show that $\#(C \cap t_y^* C) = 2$ only in the case $0 \in C \cap t_y^* C$ and $C \neq t_y^* C$.

Note first that the dual abelian variety $\widehat{J}$ parameterizes the translates of C: $\widehat{J} = \{[t_y^* C] \mid y \in J\}$. In particular $t_y^* C \neq C$ for all $y \neq 0$. Moreover, $C^2 = 2$ means that $C \cap t_y^* C$ consists of 2 points counted with multiplicities.

Now consider the closure $\widetilde{C}$ of the set $\{(C \cap t_y^* C) \setminus \{0\} \mid 0 \in C \cap t_y^* C\}$. $\widetilde{C}$ is certainly of dimension 1 contained in C and as C is irreducible, we have $\widetilde{C} = C$. Note that

$$0 \in C \cap t_y^* C \iff y \in C.$$

Hence the map $C - \{0\} \to \widetilde{C}$, $y \mapsto (C \cap t_y^* C) \setminus \{0\}$ extends to a morphism

$$\varphi : C \to \widetilde{C} = C.$$

But φ is not constant. So by Hurwitz' formula it has to be an isomorphism. This means that $(C \cap t_y^* C \setminus \{0\}) \neq 0$ if $y \neq 0$, which was to be shown. $\qquad\square$

For the rest of this section let L denote an ample line bundle of type (d_1, d_2) on the abelian surface A. By a *curve of genus* 2 we mean an irreducible reduced curve of geometric genus 2. As a consequence of Proposition 2.1 we obtain:

Proposition 2.2. *Any curve of genus 2 in the linear system $|L|$ admits at most ordinary singularities of multiplicity $\leq \frac{1}{2}(1 + \sqrt{8d_1d_2 - 7})$.*

Proof. Suppose $\overline{C} \subset |L|$ is of genus 2 and $C \to \overline{C}$ its normalization. The universal property of the Jacobian yields an isogeny $f : J = J(C) \to A$ such that after a suitable embedding $C \hookrightarrow J$ the following diagram commutes

$$
\begin{array}{ccc}
C & \subset & J \\
\downarrow & & \downarrow f \\
\overline{C} & \subset & A
\end{array}
$$

f being étale, it is clear that $\overline{C}$ cannot admit a cusp. Suppose $\overline{C}$ admit a tacnode, i.e. a point x such that $\overline{C}$ has 2 linear branches $\overline{C}_1$ and $\overline{C}_2$ touching in x.

Let x_1 and x_2 be 2 preimages of x in C. The branches C_1 and C_2 of C in x_1 and x_2 map to $\overline{C}_1$, and $\overline{C}_2$. Since $t^*_{x_1-x_2}(x_1) = x_2$ this implies that $t^*_{x_1-x_2}C$ and C intersect in x_2 of multiplicity 2. But this contradicts Proposition 2.1. Hence $\overline{C}$ admits only ordinary singularities. Suppose $\overline{C}$ admits an ordinary singularity of multiplicity ν. Then

$$
2 \leq g(\overline{C}) = d_1 d_2 + 1 - \frac{\nu(\nu - 1)}{2}
$$

since there is no curve of geometric genus ≤ 1 in $|L|$. But $\frac{1}{2}(1 \pm \sqrt{8d_1d_2 - 7})$ are the roots of the polynomial $\nu^2 - \nu - 2d_1d_2 + 2$ in ν, implying $\nu \leq \frac{1}{2}(1 + \sqrt{8d_1d_2 - 7})$. $\qquad\square$

Remark 2.3. In the situation of Prop. 2.2 there are only finitely many curves $\overline{C}$ of geometric genus 2 in $|L|$. This depends on the fact that each such curve defines an isogeny of bounded degree from the Jacobian of the normalization C of $\overline{C}$ to A and it is defined by such an isogeny up to translation. Since there are only finitely many such isogenies and also finitely many translations of $\overline{C}$ which are still in $|L|$, the conclusion follows.

§3 The incidence curve.

Let (A, L) be an abelian surface of type (d_1, d_2), where as usual we assume that L defines an irreducible polarization, i. e. the linear system $|L|$ has no fixed components.
Let $\mathbb{P}_2 \subset |L|$ be a general net in $|L|$. The *incidence curve* Γ associated to $\mathbb{P}_2$ is defined as follows.

Let $J_1(L)$ denote the first jet bundle of L. The fibre of $J_1(L)$ at a point $x \in A$ is the space $L \otimes \mathcal{O}_A/I_x^2$ of 1-jets of sections of L. It fits into the basic exact sequence

There is a natural homomorphism of sheaves

$$\sigma : \pi_2^* \mathcal{O}_{\mathbb{P}_2}(-1) \to \pi_1^* J_1(L)$$

defined by associating to every local section the constant and linear terms of its Taylor expansion. This gives a global section of the locally free sheaf

$$E = \pi_2^* \mathcal{O}_{\mathbb{P}_2}(1) \otimes \pi_1^* J_1(L).$$

By definition Γ is the vanishing scheme of this section and the third Chern class of E is the class of Γ. Note that every component of Γ has dimension at most 1 since there are at most finitely many nonreduced curves in $\mathbb{P}_2$. On the other hand, being the zero locus of a section of a rank 3 vector bundle on $A \times \mathbb{P}_2$, it has pure dimension 1 and is a local complete intersection curve. Set theoretically

$$\Gamma := \{(x, [C]) \in A \times \mathbb{P}_2 \,|\, x \quad \text{is a singular point of} \quad C\}.$$

Let π_1 and π_2 denote the natural projections of $A \times \mathbb{P}_2$. Consider the map $\varphi_{\mathbb{P}_2} : A \to \mathbb{P}_2^*$ associated to $\mathbb{P}_2$, which is a covering of degree $2d$. Here $\mathbb{P}_2^*$ denotes the dual projective space.

Lemma 3.1. $\pi_1(\Gamma)$ *is equal to the ramification locus* R *of* $\varphi_{\mathbb{P}_2}$.

Proof. If $y \in R$ then $\mathbb{P}_2(-y)$ (the pencil in $\mathbb{P}_2$ of curves passing through y) consists of curves with a fixed tangent at y: therefore $\mathbb{P}_2(-y)$ contains a curve C singular at y because such a curve satisfies only one extra linear condition. Therefore $(y, [C]) \in \Gamma$ and $\pi_1(y, [C]) = y$. $\qquad\square$

Denoting $l := c_1(\pi_1^*(L))$ and $h := c_1(\pi_2^* \mathcal{O}_{\mathbb{P}_2}(1))$ we have

$$l^2 h^2 = 2d,$$

whereas all other intersection numbers vanish. Using (1) we get

$$c_1(E) = 3(l + h) \quad \text{and} \quad \Gamma = c_3(E) = 3(l^2 h + l h^2).$$

There is an isomorphism

$$\omega_\Gamma = \omega_{A \times \mathbb{P}_2} \otimes \bigwedge^3 N_\Gamma \qquad (2)$$

Now

$$c_1(\omega_{A \times \mathbb{P}_2}|\Gamma) = -3h \cdot \Gamma = -9l^2 h^2 = -18d.$$

On the other hand $N_\Gamma = E|\Gamma$, since Γ is the top Chern class of the vector bundle E. This gives

$$c_1(N_\Gamma) = c_1(E) \cdot \Gamma = 9(l + h)(l^2 h + l h^2) = 36d.$$

So (2) yields

$$\deg \omega_\Gamma = 18d$$

and we have proven

The equality

$$\pi_2(\Gamma) = V$$

defines scheme theoretically the *discriminant locus V of the net*. Since Γ is a connected curve, we have the following:

Lemma 3.3. *V is purely one-dimensional, i.e. it is a plane curve.*

On the other hand consider the map $\varphi = \varphi_{\mathbb{P}_2} : A \to \mathbb{P}_2^*$ associated to the net $\mathbb{P}_2$. As noted above, φ is a covering of degree $2d$. Restricting to a line in $\mathbb{P}_2^*$ and using Hurwitz' formula, one sees that the branch divisor $B \subset \mathbb{P}_2^*$ of φ is a curve of degree $6d$.

From now on we will restrict to the case $(d_1, d_2) = (1,3)$. Therefore $|L|$ is a net and the branch curve B has degree 18. Let's consider the curve V. We have the following

Proposition 3.4. *$\overline{V}_{L,1}$ has degree ≤ 18 and equality holds if and only if $V = \overline{V}_{L,1}$. In this case $\pi_2 : \Gamma \to V$ is birational on each irreducible component of Γ. In particular, if Γ is reduced and $V = \overline{V}_{L,1}$ then V is also reduced.*

Proof. Let $\mathbb{P}_1 \subset |L|$ be a general pencil. The pencil $\mathbb{P}_1$ being general, it has 6 base points $x_1, \dots, x_6$. No curve in the pencil is singular at one of the base points, since otherwise $L^2 = C_1 \cdot C_2 > 6$. Let M denote the blow-up of A in the 6 base points. $f : M \to \mathbb{P}_1$ is a fibration with smooth general fibre C_{gen}. For any $s \in \mathbb{P}_1$ we denote $C_s := f^{-1}(s)$, the fibre over s. We have the following relation between the topological Euler characteristics

$$e(M) = e(C_{\text{gen}}) \cdot e(\mathbb{P}_1) + \sum_{s \in \mathbb{P}_1}(e(C_s) - e(C_{\text{gen}})).$$

(see [BPV] Proposition III.11.4). But $\sum_{s \in \mathbb{P}_1}(e(C_s) - e(C_{\text{gen}}))$ is the number of singular curves in the pencil, each counted with some multiplicity, and the nodal curves have multiplicity 1. Hence we get

$$\sum_{s \in \mathbb{P}_1}(e(C_s) - e(C_{\text{gen}})) \geq \deg \overline{V}_{L,1} \tag{3}$$

Moreover

$$\sum_{s \in \mathbb{P}_1}(e(C_s) - e(C_{\text{gen}})) = 6 - (2 - 2(4)) \cdot 2 = 18$$

since C_{gen} is smooth of genus 4. The inequality in (3) is an equality if and only if all curves in the pencil are 1-nodal and this means that $V = \overline{V}_{L,1}$. In this case a general element of any irreducible component of V is 1-nodal, and therefore π_2 is birational on each component of Γ. $\square$

§4 Abelian surfaces of type $(1,3)$, a special case.

Let (A, L) be an abelian surface of type $(1,3)$, so the linear system $|L|$ is a plane $\mathbb{P}_2 =$

ramified over a curve $B \subset \mathbb{P}_2^*$ of degree 18. Our aim is to understand the curves V and B for a general abelian surface (A, L). In this section we study first a special case.

Let E be an elliptic curve and consider the abelian surface $A = E \times E$. Let L denote the line bundle

$$L = \mathcal{O}_A(E \times \{0\} + \{0\} \times E + \widetilde{\triangle})$$

where $\widetilde{\triangle} = \{(x, -x) \in E \times E\}$ denote the antidiagonal. According to [BL1] L defines an irreducible polarization of type $(1, 3)$. Moreover, in [BL2] Proposition 3.3 it is shown that the branch divisor $B \subset \mathbb{P}_2^*$ of $\varphi_L : A \to \mathbb{P}_2^*$ is

$$B = 3D \tag{1}$$

where D is a plane sextic with with 9 cusps, the dual of the elliptic curve E, considered as a plane cubic in $\mathbb{P}_2$ embedded by $|3 \cdot 0|$ (see [BL2]). We first determine the ramification divisor $R \subset A$ of φ_L.

We may assume that L is a symmetric line bundle. Then the extended Heisenberg group $H(L)^e$ acts on B. We may choose the coordinates $(x_0 : x_1 : x_2)$ of $\mathbb{P}_2^*$ in such a way that $H(L)^e$ is generated by (see [CAV])

$$\sigma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & & \\ & \rho & \\ & & \rho^2 \end{pmatrix} \quad \text{and} \quad \iota = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

where $\rho = \exp(\frac{2\pi i}{3})$. Consider the set

$$\mathrm{Fix} H(L)^e = \{x \in \mathbb{P}_2^* \,|\, g(x) = x \text{ for some } 1 \neq g \in H(L)^e\}$$

of points in $\mathbb{P}_2^*$ having non trivial stabilizer. It consists of 9 lines and 9 points

$$l_{1i} = \{x_1 = \rho^i x_0\}, \qquad l_{2i} = \{x_2 = \rho^i x_1\}, \qquad l_{3i} = \{x_0 = \rho^i x_2\}$$
$$y_{1i} = (1 : -\rho^i : 0), \qquad y_{2i} = (0 : 1 : -\rho^i), \qquad y_{3i} = (-\rho^i : 0 : 1)$$

with $i = 0, 1, 2$. Moreover, the 9 lines intersect in triples in 12 points $x_1, \ldots, x_{12}$, say.

This immediately gives that if $x \in l_i$, $x \neq x_j$ for all j or if $x = y_i$ for some i, then the orbit of x consists of 9 elements. The orbit of x_i ($i = 1, \ldots, 12$) consists of 3 elements. To be more precise, there are exactly 4 orbits of order 3, namely

$$\begin{aligned}
O(1 : 0 : 0) &= \{(1 : 0 : 0), \quad (0 : 1 : 0), \quad (0 : 0 : 1)\} \\
O(1 : 1 : 1) &= \{(1 : 1 : 1), \quad (1 : \rho : \rho^2), \quad (1 : \rho^2 : \rho)\} \\
O(1 : 1 : \rho) &= \{(1 : 1 : \rho), \quad (1 : \rho : 1), \quad (\rho : 1 : 1)\} \\
O(1 : 1 : \rho^2) &= \{(1 : 1 : \rho^2), \quad (1 : \rho^2 : 1), \quad (\rho^2 : 1 : 1)\}
\end{aligned}$$

All other orbits of $H(L)^e$ in $\mathbb{P}_2^*$ consist of 18 points.

According to [BL2] Proposition 1.1 with respect to the coordinates $y_0 = 3x_0^2 - 3\lambda x_1 x_2$, $y_1 = 3x_1^2 - 3\lambda x_0 x_2$, $y_2 = 3x_2^2 - 3\lambda x_0 x_1$ the sextic D is given by the equation

$$(y_0^6 + y_1^6 + y_2^6) + 2(2\lambda^3 - 1)(y_0^3 y_1^3 + y_1^3 y_2^3 + y_0^3 y_2^3)$$

Here $\lambda \in \mathbb{Q} - \{1, \rho, \rho^2\}$ is a parameter depending on the elliptic curve E. It can be explicitly determined from the j-invariant of E. Now an immediate computation using the above equation gives

Lemma 4.1. *For a general elliptic curve E the branch curve B does not contain an orbit of order 3 under the extended Heisenberg group $H(L)^e$.*

Denote by $\triangle := \{(x, x) \in E \times E\}$ the diagonal, $\Gamma_{-2} = \{(x, -2x) \in E \times E\}$ the graph of $(-2)_E$ and $\Gamma^t_{-2} = \{(-2x, x) \in E \times E\}$ its transpose. Then we have

Proposition 4.2. (a): *The ramification divisor of $\varphi_L : E \times E \to \mathbb{P}_2^*$ is*

$$R = \triangle + \Gamma_{-2} + \Gamma^t_{-2}.$$

(b) $\varphi_L|\triangle : \triangle \to D$ *coincides with the duality map $\triangle = E \to E^*$, $x \mapsto$ tangent at x. In particular $\varphi_L|\triangle$ is bijective.*
(c) $p_a(R) = 28$.

Proof. The map $\varphi_L : E \times E \to \mathbb{P}_2^*$ is a Galois covering with Galois group D_3 generated by

$$T = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [BL2] Corollary 3.2). Since $J|\triangle = \mathrm{id}$, the map φ_L is ramified in $\triangle$. Hence

$$\triangle + \Gamma_{-2} + \Gamma^t_{-2} \subseteq R$$

since $T(\triangle) = \Gamma_{-2}$ and $T^2(\triangle) = \Gamma^t_{-2}$.
On the other hand $\varphi_L|\triangle$ is given by a 2-dimensional sublinear system of

$$
\begin{aligned}
L|\triangle &= \mathcal{O}_\triangle((E \times \{0\} + \{0\} \times E + \widetilde{\triangle}) \,|\, \triangle) \\
&= \mathcal{O}_\triangle(3(0,0) + (\alpha, \alpha) + (\beta, \beta) + (\gamma, \gamma)) \\
&\quad \text{where} \quad \{0, \alpha, \beta, \gamma\} \quad \text{are the division points of} \quad \triangle. \\
&= \mathcal{O}_\triangle(6(0,0)), \quad \text{since} \quad \alpha + \beta + \delta \sim 3 \cdot 0 \text{ in } \triangle.
\end{aligned}
$$

As $\varphi_L(\triangle) = D$ is just the dual curve of $\triangle = E$, this implies (b) and hence (a), since $B = 3D$. As for (c), $R^2 = 54$ implies $p_a(R) = 28$. $\qquad \square$

Using this one can show

Proposition 4.3. *Let $V \subset \mathbb{P}_2$ denote the discriminant curve of $(E \times E, \mathcal{O}_{E \times E}(E \times \{0\} + \{0\}E + \widetilde{\triangle}))$. Then*

$$V = 3E + l_1 + \ldots + l_9$$

where E denotes the elliptic curve as a plane cubic as above and $l_1, \ldots, l_9$ are the flex tangents of E.

Proof. The projections π_1 and π_2 of $A \times \mathbb{P}_2$ and the map $\varphi_L : A \to \mathbb{P}_2^*$ restrict to the following diagram

$$\Gamma$$
$$\pi_1 \swarrow \qquad \qquad \searrow \pi_2$$
$$R \qquad \qquad V$$

where D is the plane sextic with 9 cusps dual to E. Let

$$V = V_0 + l_1 + \ldots + l_v$$

with lines $l_1, \ldots, l_v$ and V_0 containing no line.

We claim that V contains exactly 9 lines and these are the flex tangents of red $V_0 = E$. To see this note that for any line $k \subset \mathbb{P}_2^*$ passing through a cusp of D the branch divisor of $\varphi|\varphi^{-1}(k) : C = \varphi^{-1}(k) \to k$ contains a point of multiplicity ≥ 6, hence C is singular (because φ_L has degree 6). But the pencils of lines through the cusps of D correspond just to the flex tangents $l_1 + \ldots + l_9$ of the dual curve.

On the other hand according to Propositions 3.2 and 4.2 (c) the curves Γ and R have the same arithmetic genus 28. Hence $\pi_1 : \Gamma \to R$ is an isomorphism apart from possibly contracting lines of Γ to a point in R. Applying Proposition 4.2 (a) we conclude that Γ contains 3 copies of the elliptic curve E which map under π_2 to V. This implies $3E + l_1 + \ldots + l_9 \subseteq V$.

It is not difficult to describe the 1-dimensional family of singular curves parametrized by $3E \subset V$. The surface A contains a 3-dimensional family of singular curves of the form:

$$E \times \{a\} + \{b\} \times E + t_{(c,c)}^* \widetilde{\triangle}$$

with $a, b, c \in E$. The general curve of this family is easily seen to have 3 nodes. Intersecting with $|L|$ we get a 1-dimensional family of singular curves in $|L|$ the general one having 3 nodes. This family is parametrized by the plane cubic $3E \subset V$.

Consider now a general pencil $\mathbb{P}_1 \subset |L|$. The computation made in the proof of Prop. 3.4 gives in this case a contribution of 3 from each curve belonging to $3E$, because each such curve is trinodal. But then since the number we obtain is 18, and this is also the degree of $3E + l_1 + \ldots + l_9$ we conclude that $V = 3E + l_1 + \ldots + l_9$ as asserted. $\qquad\square$

Remark 4.4. The proof of Prop. 4.3 shows that $\overline{V}_{L,1} = l_1 + \ldots + l_9$; in particular $V \neq \overline{V}_{L,1}$ in this case. In fact the proof shows that the curves of $\mathbb{P}_1 \cap (l_1 + \ldots + l_9)$ contribute with multiplicity one in the computation made in Prop. 3.4, and this can only be possible if these curves are 1-nodal.

Note that the curve parametrized by a general point of $3E$ is not in $V_{L,1}$, but it is nevertheless a nodal curve (it is in $V_{L,3}$). In particular the general curve of each component of V is a nodal curve.

Finally we remark that from the analysis of this section it follows that the incidence curve Γ of the special surface $(E \times E, L)$ considered is reduced.

§5 The general abelian surface of type $(1, 3)$.

Let (A, L) be a general abelian surface of type $(1, 3)$. Let $V \subset \mathbb{P}_2 = |L|$ and $B \subset \mathbb{P}_2^*$ again denote the discriminant curve of L and the branch curve of $\varphi_L : A \to \mathbb{P}_2^*$. After showing that V and B are irreducible and reduced, we compute their singularities.

Proof. Since (A, L) is general we may assume that every element of $|L|$ is irreducible. Every curve belonging to V is irreducible of geometric genus ≤ 3 because the genus of the nonsingular elements of $|L|$ is 4. Since there are only finitely many curves of geometric genus 2 in $|L|$ (Remark 2.3) we deduce that each irreducible component of V consists generically of curves of geometric genus 3, which therefore either have one node or one ordinary cusp, and no other singularity. Let's consider a family of polarized abelian surfaces of type $(1, 3)$ with general fibre (A, L) and special fibre $(E \times E, L_0)$ with $L_0 = \mathcal{O}_{E \times E}(E \times \{0\} + \{0\} \times E + \widetilde{\triangle})$. Under this degeneration the curve V specializes to a curve V' contained in the discriminant curve of $|L_0|$. A component of V consisting generically of cuspidal curves would degenerate to a component of V' having the same property. But we have seen (Remark 4.4) that a general element of any component of this curve parametrizes nodal curves. So each irreducible component of V consists generically of 1-nodal curves, i.e. $V = \overline{V}_{L,1}$.

To prove the last assertion note that under the above degeneration the incidence curve Γ of (A, L) degenerates to the incidence curve of $(E \times E, L_0)$, which is reduced (Remark 4.4). Therefore Γ is reduced as well. The conclusion now follows from Proposition 3.4. $\square$

Proposition 5.2. *For a general abelian surface (A, L) the branch divisor $B \subset \mathbb{P}_2^*$ is a reduced and irreducible curve of degree* 18.

Proof. Since $\varphi_L(R) = B$ it suffices to show that the ramification divisor $R \subset A$ is reduced and irreducible and that $\varphi_L : R \to B$ is birational. As in the proof of Proposition 5.1 consider a family of polarized abelian surfaces of type $(1, 3)$ with general fibre (A, L) and special fibre $(E \times E, L_0)$ with $L_0 = \mathcal{O}_{E \times E}(E \times \{0\} + \{0\} \times E + \widetilde{\triangle})$. Certainly here the ramification divisor R of $\varphi : A \to \mathbb{P}_2^*$ specializes to the ramification divisor R_0 of $\varphi_{L_0} : E \times E \to \mathbb{P}_2^*$. According to Proposition 4.2 (a)

$$R_0 = \triangle + \Gamma_{-2} + \Gamma^t_{-2}.$$

Now $R^2 = R_0^2 = 54$. This implies that $\mathcal{O}_A(R)$ is of type $(1, 27)$ or $(3, 9)$. The abelian surface (A, L) being general, it cannot be of type $(1, 27)$. Hence

$$\mathcal{O}_A(R) \equiv L^3.$$

If R were not irreducible and reduced, say $R = R_1 + R_2$ or $R_1 + R_2 + R_3$, then we would have $\mathcal{O}_A(R_1) \equiv L$. But R_1 specializes to one of the curves $\triangle, \Gamma_{-2}$ or Γ^t_{-2}, say to $\triangle$. This would give $\mathcal{O}_{E \times E}(\triangle) \equiv L_0$, a contradiction, since $L_0^2 = 6$ and $\triangle^2 = 0$.
It remains to be proved that $\varphi_L : R \to B$ is birational. For any point $p \in \mathbb{P}_2^*$ we have that $\varphi^{-1}(p)$ consists of the 6 base points (counted with multiplicity) of a pencil in $|L|$ which is the line of $\mathbb{P}_2$ dual to p. Since $B = \varphi(R) = \varphi(\pi_1(\Gamma))$, we have that $p \in B$ if and only if the corresponding pencil is of the form $|L(-x)|$ for some $(x, [C]) \in \Gamma$, (and of course $\varphi(x) = p$). If p is a general point of a component of B then $[C] \in V_{L,1}$ and x is the node of C (by Prop. (5.1)); moreover the pencil $|L(-x)|$ is the tangent line to $V_{L,1}$ at $[C]$ (see §1). Assume that $\varphi_L : R \to B$ is not birational. Then for a general choice of $p \in B$ there are $(x_1, [C_1]), (x_2, [C_2]) \in \Gamma$ such that $p = \varphi(x_1) = \varphi(x_2)$ and $x_1 \neq x_2$. If $C_1 = C_2 = C$ then the curve C has two nodes and therefore it is not in $V_{L,1}$, a contradiction. If $C_1 \neq C_2$ then the line $|L(-x_1)| = |L(-x_2)|$ of $\mathbb{P}_2$ is bitangent to $V = \overline{V}_{L,1}$ at the points $[C_1]$ and

Proposition 5.3. *The curve V of a general abelian surface (A, L) of type $(1, 3)$ is reduced of degree 18 and admits at most nodes, cusps or ordinary tacnodes as singularities.*

Proof. By 5.1 and 3.4 V is reduced of degree 18. If $[X] \in V$ then, thanks to Proposition 2.2, one of the following occurs:

    i) X has one node $(g = 3)$

    ii) X has one cusp $(g = 3)$

    iii) X has two nodes $(g = 2)$

and there are no other possibilities.

    Case i) occurs if and only if $[X] \in V_{L,1}$. In this case $[X]$ is a nonsingular point of V.

    Suppose we are in case ii). The natural map $h : \frac{H^0(L)}{\langle \sigma_X \rangle} \to H^0(T_X^1)$ (see §1) is bijective: in fact domain and codomain are both of dimension 2 and $Im(h)$ contains two independent vectors corresponding to an infinitesimal deformations which smooths X and to an infinitesimal deformation which deforms X to a nodal curve. Therefore V induces, locally around $[X]$, a semiuniversal deformation of the cusp. It is well known that the locus parametrizing singular fibres is an ordinary cusp at $[X]$ (see e.g. [D-H], p. 3).

    Suppose finally that we are in case iii). By taking a general pencil $\mathbb{P}_1 \subset |L|$ containing $[X]$ and applying the same argument of Proposition 3.4 we see that $[X]$ absorbs two of the 18 intersections $\mathbb{P}_1 \cap V$, because $e(X) = e(X_{gen}) + 2$. Therefore $[X]$ is a double point of V. Let N_1, N_2 be the nodes of X. For each $i = 1, 2$ the vector space

$$\frac{H^0(\mathcal{I}_{N_i} \otimes L)}{\langle \sigma_X \rangle} \subset \frac{H^0(L)}{\langle \sigma_X \rangle}$$

is the tangent space to the local analytic family of deformations of X which keep the node N_i. Each such local family is a linear branch of V at $[X]$ (see [D-H] again). Therefore V has two linear branches at $[X]$, which is therefore either an ordinary node or an ordinary tacnode. The last possibility occurs precisely when $H^0(\mathcal{I}_{N_1} \otimes L) = H^0(\mathcal{I}_{N_2} \otimes L)$, and this happens if and only if N_1 and N_2 are in the same fibre of φ_L. $\qquad \square$

Proposition 5.4. *Let (A, L) be a general abelian surface of type $(1, 3)$ and let $V = V' + l_1 + \cdots + l_v$, where $l_1, \ldots, l_v$ are the line components of V. Then B and V' are dual to each other (in particular V' is irreducible).*

Proof. As seen in the proof of Prop. 5.2, for any point $p \in \mathbb{P}_2^*$ we have that $\varphi^{-1}(p)$ consists of the 6 base points (counted with multiplicity) of a pencil in $|L|$ which is the line of $\mathbb{P}_2$ dual to p if p is a general point of a component of B the curve C is nodal at x, $[C] \in V'$ and the pencil $|L(-x)|$ is the tangent line to V' at $[C]$. This proves that $B = V'^\vee$, the dual curve of V'. The converse follows from the fact that $V'^{\vee\vee} = V'$. $\qquad \square$

Recall that the projections π_1, and π_2 and the map φ_L restrict to the following diagram

$$
\begin{array}{ccc}
& \Gamma & \\
{\pi_1} \swarrow & & \searrow {\pi_2} \\
R & & V \\
{\varphi_L} \downarrow & & \\
B & &
\end{array}
$$

on R. According to Lemma 4.1 we may assume that the number v of lines is divisible by 9, since (A, L) is general. As B is dual to V', we are left with 2 cases .

I. $V = V'$ irreducible of degree 18 in $\mathbb{P}_2$ and $v = 0$

II. $V = V' + l_1 + \ldots + l_9$ with V' irreducible of degree 9 in $\mathbb{P}_2$.

For the curve Γ this implies: Either $\Gamma = \Gamma'$ is irreducible or $\Gamma = \Gamma' + l'_1 + \ldots + l'_9$ with Γ' irreducible and $\pi_1(l'_i)$ is a point in R for $i = 1, \ldots, 9$.

Proposition 5.5. *The curve Γ' is smooth.*

Proof. By Proposition 5.3 V' and hence also Γ' admit at most nodes, cusps and ordinary tacnodes.

If Γ' has a node or an ordinary tacnode in $(x, [C])$, Then V' also has a node or a tacnode in $[C]$. This implies that the curve C is of genus 2. But C admits at most nodes as singularities by Proposition 2.2. Hence $\pi_2^{-1}([C])$ consists of 2 points, so both of them have to be smooth, a contradiction.

If Γ' has a cusp in $(x, [L])$, then V' has a cusp in $[C]$. But Γ' projects birationally onto the dual curve B of V' and the dual of a cusp is a smooth point. Hence $\varphi_L(x)$ is a smooth point of B and so is $(x, [C])$ in Γ', since all maps are birational. $\qquad\square$

Now we are in a position to prove the main theorem of this paper.

Theorem 5.6. *Let (A, L) be a general abelian surface of type $(1, 3)$, $B \subset \mathbb{P}_2^*$ the branch locus of $\varphi_L : A \to \mathbb{P}_2^*$ and $V \subset \mathbb{P}_2 = |L|$ the closure of the Severi variety $V_{L,1}$.*

Both V and B are irreducible of degree 18 in $\mathbb{P}_2$ (resp. $\mathbb{P}_2^*$) smooth apart from 72 cusps and 36 nodes or tacnodes (each tacnode counting as two nodes), and admit 72 flexes and 36 bitangents.

Proof. By Proposition 5.3 V admits at most nodes, cusps or ordinary tacnodes as singularities. Since for the following computations an ordinary tacnode counts exactly the same as 2 nodes, we assume for simplicity that V admits at most nodes and cusps.

Let V' the irreducible component of V as above. Denote by ν (respectively κ, f, and b) the number of nodes (respectively cusps, flexes and bitangents) of V'. The classical Plücker formulas (see [Wa]) say that these numbers are related to the degree d of V' and the degree m of its dual plane curve B as follows:

$$m = d(d - 1) - 2\nu - 3\kappa \tag{1}$$
$$d = m(m - 1) - 2b - 3f \tag{2}$$
$$f = 3d(d - 2) - 6\nu - 8\kappa \tag{3}$$

Suppose first that we are in case II: $V = V' + l_1 + \ldots + l_9$. Let $\Gamma = \Gamma' + l'_1 + \ldots l'_9$ denote the corresponding decomposition. Then Γ' is smooth by Proposition 5.5 and by Proposition 3.2 we have

$$p_a(\Gamma) = p_a(R) = 28.$$

For this situation there are only 2 possible cases:

(i) The lines $l_1, \ldots, l_9$ intersect Γ' transversally in 1 point each and $\pi_2 : \Gamma' \to R$ is an

(ii) The line l_i intersects Γ' transversally in 2 points x_i and x'_i for $i = 1, \ldots, 9$ and $\pi_2(x_i) = \pi_2(x'_i)$ is a node in R whereas $\pi_2 : \Gamma' \to R$ is an isomorphism elsewhere (here we use the fact that the curve R is Heisenberg invariant and that each $H^0(L)^e$-orbit consists of ≥ 9 points).

Suppose first we are in case (i): Then $g(V') = g(\Gamma') = g(R) = 28$. On the other hand V' is a plane curve of degree 9. Hence

$$28 = p_a(V') = g(V') + \nu + \kappa = 28 + \nu + \kappa,$$

so $\nu = \kappa = 0$. But then (1) says $18 = 72$, a contradiction. In the second case $g(V') = g(\Gamma) = p_a(R) - 9 = 19$. Hence

$$28 = p_a(V') = 19 + \nu + \kappa.$$

So $\nu + \kappa = 9$. On the other hand (1) says

$$2\nu = 3\kappa = 54.$$

This implies $\kappa = 18, \nu = 27$, a contradiction.

Hence $V = V'$ is irreducible of degree 18 in $\mathbb{P}_2$. So Γ is the normalization of V and $g(Y) = g(\Gamma) = 28$ and we have

$$136 = p_a(V) = g(V) + \nu + \kappa = 28 + \nu + \kappa$$

i. e.
$$\nu + \kappa = 108.$$

On the other hand (1) says
$$2\nu + 3\kappa = 288.$$

Combining both equations we obtain $\kappa = 72$ and $\nu = 36$. But then (2) and (3) give $f = 72$ and $b = 36$.

By Proposition 5.4 the statement follows also for B. □

Remark. The 36 nodes of V correspond to the curves of genus 2 in $|L|$, which are all 2-nodal. The 72 cusps correspond to the curves of genus 3 in $|L|$ having a cusp. We don't know whether tacnodes actually occur for general (A, L) (actually for any (A, L)).

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